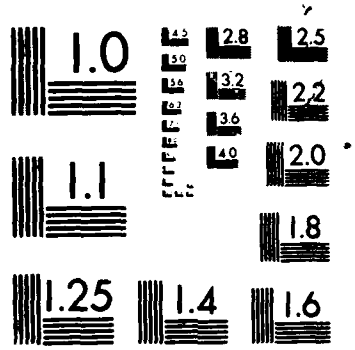
AD-A186 010 EXPLICIT SOLUTIONS OF MOMENT PROBLEMS 1(U) NORTH CAROLINA UNIV AT CHAPEL HILL CENTER FOR STOCHASTIC PROCESSES I KUZNEZOVA-SHOLPO ET AL. JUL 87 TR-197 1/1

UNCLASSIFIED AFOSR-TR-87-1146 F49620-85-C-0144 F/G 12/3 NL



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS 1963 A

Unclassified
SECURITY CLASSIFICATION OF THIS PAGE

AD-A186 018

DTIC FILE COPY

REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION: Unclassified	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY	3. DISTRIBUTION/AVAILABILITY OF REPORT: Approved for public release; distribution unlimited.
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE	
4. PERFORMING ORGANIZATION REPORT NUMBER(S): Technical Report No. 197	5. MONITORING ORGANIZATION REPORT NUMBER(S): AFOSR-TR- 87-1146
6a. NAME OF PERFORMING ORGANIZATION: University of North Carolina; 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION: AFOSR/NM
6c. ADDRESS (City, State, and ZIP Code): Statistics Dept., 321-A Phillips Hall 039-A, Chapel Hill, NC 27514	7b. ADDRESS (City, State, and ZIP Code): AFOSR/NM, Bldg 410, Bolling AFB DC 20332-6448
8a. NAME OF FUNDING/SPONSORING ORGANIZATION: AFOSR; 8b. OFFICE SYMBOL (If applicable): NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: AFOSR Grant No. F49620 85C 0144
8c. ADDRESS (City, State, and ZIP Code): AFOSR/NM, Bldg 410, Bolling AFB DC 20332-6448	10. SOURCE OF FUNDING NUMBERS: PROGRAM ELEMENT NO. 61102F; PROJECT NO. 2304; TASK NO. A-5; WORK UNIT ACCESSION NO.

DTIC ELECTE OCT 0 7 1987 S E

11. TITLE (Include Security Classification): Explicit solutions of moment problems I

12. PERSONAL AUTHOR(S): Dr. Irina Kuznezova-Sholpo and Dr. Svetlozar T. Rachev

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
preprint	FROM 9/86 TO 8/87	July 1987	25

16. SUPPLEMENTARY NOTATION

17. COSATI CODES			18. SUBJECT TERMS
FIELD	GROUP	SUB-GROUP	Key Words and Phrases: Moment problem, explicit solutions, dual representations, marginal problem, minimal metric.

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

The moment problems

minimize (maximize) $\{E\|X_1 - X_2\|^p : E\|X_i\|^{q_j} = a_{ij},\ i=1,2,\ j=1,\dots,n\}$, where X_i takes values in a separable norm space are investigated. Explicit solutions when $n=1$ are given for all $p \geq 0$ and $q_j \geq 0$ $(0^0 := 0)$. If $n=2$, the minimization problem is solved for $1 \leq q_1 \leq p < q_2$ and the maximization problem for $0 < p \leq q_1$, $1 \leq q_1 < q_2$ or $0 < q_1 < q_2 < p$. Possible generalizations and open problems are presented.

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT: ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION: Unclassified
22a. NAME OF RESPONSIBLE INDIVIDUAL: Maj. Brian Woodruff	22b. TELEPHONE (Include Area Code): (202) 767-5026; 22c. OFFICE SYMBOL: NM

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE
Unclassified

CENTER FOR STOCHASTIC PROCESSES

AFOSR-TR- 87-1146

Department of Statistics
University of North Carolina
Chapel Hill, North Carolina

EXPLICIT SOLUTIONS OF MOMENT PROBLEMS I

by

Irina Kuznezova-Sholpo

and

Svetlozar T. Rachev

Technical Report No. [redacted] 197

July 1987

EXPLICIT SOLUTIONS OF MOMENT PROBLEMS I

by

Irina Kuznezova-Sholpo
Saratov State University
Saratov, USSR

and

Svetlozar T. Rachev
Institute of Mathematics
Bulgarian Academy of Sciences
Sofia, Bulgaria

and

Center for Stochastic Processes
Statistics Department
University of North Carolina
Chapel Hill, NC 27514

Abstract

The moment problems

minimize (maximize) $\{E\|X_1 - X_2\|^p : E\|X_i\|^{q_j} = a_{ij},\ i=1,2,\ j=1,\ldots,n\}$, where X_i takes values in a separable norm space are investigated. Explicit solutions when $n=1$ are given for all $p \geq 0$ and $q_j \geq 0$ $(0^0 := 0)$. If $n=2$, the minimization problem is solved for $1 \leq q_1 \leq p < q_2$ and the maximization problem for $0 < p \leq q_1$, $1 \leq q_1 < q_2$ or $0 < q_1 < q_2 < p$. Possible generalizations and open problems are presented.

AMS 1980 Subject Classification: Primary 60E05, 60E15; Secondary 28A35, 28C05.

Keywords and Phrases: Moment problem, explicit solutions, dual representations, marginal problem, minimal metric.

*This research was supported in part by the Air Force Office of Scientific Research under Contract No. F49620 85 C 0144 while the second named author was visiting the Center for Stochastic Processes, University of North Carolina, Chapel Hill, NC.

1. Introduction

In probability theory the following two measure theoretic problems are well known (see e.g. Levin and Miljutin (1978), Kellerer (1984), Kemperman (1983), Haneveld (1985), Rüschendorf (1985), Rachev (1984) and references there):

A. Marginal problem: *For fixed probability measures (laws)* P_1 *and* P_2 *on a measurable space* U *and a measurable function* c *on the product space* $U^2 = U \times U$:

(1.1) minimize (maximize) $\int_{U^2} c(x,y)P(dx,dy)$,

where the laws P *on* U^2 *have marginals* P_1 *and* P_2, *i.e.*

(1.2) $\pi_i P = P_i$, $i = 1,2$,.

B. Moment problem: *For fixed real numbers* a_{ij} *and real-valued continuous functions* f_{ij} $(i = 1,2,\ j = 1,\ldots,n)$.

(1.3) minimize (maximize) $\int_{U^2} c(x,y)P(dx,dy)$,

where the law P *on* U^2 *satisfies the marginal moment conditions.*

(1.4) $\int_U f_{ij} dP_i = a_{ij}$, $i = 1,2$, $j = 1,\ldots,n$.

Dual relationships and explicit solutions of the marginal problem for different spaces U and criterion functions c are given by Levin and Miljutin (1978), Kellerer (1984), Haneveld (1985), Cambanis, Simons and Stout (1976), Cambanis and Simons (1982), Rüschendorf (1985) and Rachev (1984, 1985). In citing some results concerning duality and explicit solutions of A, we shall use the following notations:

(1.5) (U,d) is a separable metric space with metric d;

(1.6) $\mathcal{P}(U^k)$ is the space of all Borel probability measures on the Cartesian product U^k;

(1.7) $\mathcal{H}$ is the class of all convex functions

$$H : [0,\infty) \to [0,\infty), H(0)=0, \; K_H := \sup_{t\geq 0} H(2t)/H(t) < \infty;$$

(1.8) $$\mathcal{P}_H := \{P \in \mathcal{P}(U) : \int_U H(d(x,a))P(dx) < \infty\}, \; H \in \mathcal{H};$$

(Note that $\mathcal{P}_H$ does not depend on $a \in U$ for any $H \in \mathcal{H}$);

(1.9) $D(x,y) := H(d(x,y))$;

(1.10) $$\mathrm{Lip}(U) := \{f : U \to \mathbb{R}, \sup_{x\in U}|f(x)| < \infty, \exists\, \alpha(f)>0 : |f(x)-f(y)| \leq \alpha(f)d(x,y)$$

$$\text{for all } x,y \in U\};$$

(1.11) $\underline{G}_H := \{(f,g) : f,g \in \mathrm{Lip}(U),\; f(x)+g(y) \leq D(x,y),\; x,y \in U\}$;

(1.12) $\overline{G}_H := \{(f,g) : f,g \in \mathrm{Lip}(U),\; f(x) \geq 0,\; g(y) \geq 0,\; f(x)+g(y) \geq D(x,y),\; x,y \in U\}$.

<u>Theorem A</u>. i) (Duality solutions of A). *Let* (U,d) *be a separable metric space* $H \in \mathcal{H}$, $P_1, P_2 \in \mathcal{P}_H$. *Then*

(1.13) $$\hat{L}_H(P_1,P_2) := \inf\{\int_{U^2} D(x,y)P(dx,dy) : P \in \mathcal{P}(U^2),\; \pi_i P = P_i,\; i=1,2\}$$

$$= \sup\{\int_U f dP_1 + \int_U g dP_2 : (f,g) \in \underline{G}_H\}$$

and

(1.14) $$\check{L}_H(P_1,P_2) := \sup\{\int_{U^2} D(x,y)P(dx,dy) : P \in \mathcal{P}(U^2),\; \pi_i P = P_i,\; i=1,2\}$$

$$= \inf\{\int_U f dP_1 + \int_U g dP_2 : (f,g) \in \overline{G}_H\}.$$

(ii) (Explicit solutions of A). *If* $(U,d) = (\mathbb{R}, |\cdot|)$ *and* H *is a convex function then*

$$(1.15)\qquad \hat{L}_H(P_1,P_2) = \int_0^1 D(F_1^{-1}(t),\ F_2^{-1}(t))dt,$$

$$(1.16)\qquad \check{L}_H(P_1,P_2) = \int_0^1 D(F_1^{-1}(t),\ F_2^{-1}(1-t))dt,$$

where F_j *is the distribution function corresponding to* P_j *and* F_j^{-1} *is its inverse,* $j=1,2$.

Indication: i) see Rachev (1985). ii) see Cambanis, Simons and Stout (1976).

Kellerer (1984) provides duality solutions of A for general c; however, in case $c=D$, his dual solutions are not as sharp as these in i).

The possible solutions of the marginal problems are related to the dual and explicit expressions for the so-called *minimal metrics* (see Zolotarev (1976,1983)) and *maximal distances* (see Rachev (1985)) that are fruitful in the development of a considerable range of stability problems for stochastic models (see Zolotarev (1983), Rachev (1984)).

Owing to the number of its important applications (see Shohat and Tamarkin (1943), Ahiezer and Krein (1962), Karlin and Studden (1966), Hoeffding (1955), Hoeffding and Shrikhande (1956), Basu and Simons (1983), Kemperman (1972,1983), Haneveld (1985)) the moment problem can also be treated as an approximation of the marginal problem. Indeed, if the laws P_1 and P_2 (see (1.2)) are not determined completely and if only some functionals of P_1 and P_2 are given (see (1.4)) then one has to solve the problem B instead of A. The significance of the moment problem for the theory of probability metrics was also stressed by Sholpo (1983) and Rachev (1985).

General dual representations of moment problems on a compact space U are given in Kemperman (1972,1983), Haneveld (1985). In a more general case of a completely regular topological space U, dual expressions are given in Kemperman (1983) under a "tightness" condition on the pairs (f_{ij}, a_{ij}) $(i=1,2,\ j=1,\dots,n)$.

The present paper is devoted to the explicit solutions of some moment problems on separable metric space U with metric d. In this case, considering a "rich enough" probability space $(\Omega, A, \Pr)$ without atoms and the space $X = X(U)$ of all U-valued random variables (rv's) X on $(\Omega, A, \Pr)$ one can rewrite the moment problem B as follows:

$$\text{(1.17)}\quad \text{minimize(maximize)}\{\mathbb{E}\, c(X_1, X_2) : \mathbb{E} f_{ij}(X_i) = a_{ij},\ \ i=1,2,\ j=1,2,\dots,n\}.$$

In fact, the above assumptions guarantee that the set of all Borel probability measures on U^2 coincides with the set of all joint distributions $\Pr_{X,Y}$ of pairs of rv's X,Y (see Rachev and Shortt (1987)). The main reason for considering not arbitrary but separable space (U,d) is that we need the measurability of d. For example, if $c(x,y) = d(x,y)$ in (1.5) and (U,d) is a metric space of cardinality $> \mathfrak{c}$, then the metric $d: U \times U \to \mathbb{R}$ is not measurable with respect to the product σ-algebra $\mathcal{B}(U) \times \mathcal{B}(U)$ ($\mathcal{B}(U)$ is the Borel σ-algebra on (U,d)), see Billingsley (1968).

In Section 2 explicit solutions are given for the problem (1.17) in the case where $n=1$, U is a separable norm space with norm $\|\cdot\|$ and

$$\text{(1.18)}\quad c(x,y) = h(\|x-y\|),\ \ f_{i1}(x) := g(\|x\|),\ \ i=1,2\ .$$

In particular, for any $p \ge 0$ and $q \ge 0$ we solve (1.17) with

(1.19) $c(x,y) = \|x-y\|^p,\ f_{i1}(x) = \|x\|^q.$

(Here and in the sequel 0^0 means 0).

In Section 3 we assume that U is again separable norm space but $n \geq 2$. In this case, among other results, explicit solutions are given for the moment problem (1.17) with

(1.20) $c(x,y) = \|x-y\|^p,\ f_{ij}(x) = \|x\|^{q_j},\ i=1,2,\ j=1,2,$

for $1 \leq q_1 \leq p < q_2$ when minimizing in (1.17) and
for $0 < p \leq q_1$, $1 \leq q_1 < q_2$ or $0 < q_1 < q_2 < p$ when maximizing in (1.17).
Here, we also give the explicit solution of the well known moment problem.

(1.21) $\text{minimize(maximize)}\{\mathbb{E}\|X\| : \mathbb{E}\|X\|^{q_i} = a_i,\ i=1,2,\}$

for all nonnegative p, q_1 and q_2.

In Section 4 we apply the results of Section 3 to obtain precise bounds for $\hat{L}_H(P_1,P_2)$ and $\check{L}_H(P_1,P_2)$ when the moments $\int\|x\|^q P_i(dx)$ $(i=1,2)$ are fixed. Some open problems are offered.

2. Moment problems with one fixed pair of marginal moments

Let U be a separable norm space with norm $||\cdot||$ and M be the class of all strictly increasing continuous functions $f:[0,\infty]\to[0,\infty]$, $f(0)=0$, $f(\infty)=\infty$. In the present section we treat the explicit representations of the following extremal functionals:

$$(2.1)\quad I(h,g;a,b) := \inf\{\mathbb{E}h(||X-Y||): X,Y\in \mathfrak{X}(U), \mathbb{E}g(||X||)=a, \mathbb{E}g(||Y||)=b\}$$

and

$$(2.2)\quad S(h,g;a,b) := \sup\{\mathbb{E}h(||X-Y||): X,Y\in \mathfrak{X}(U), \mathbb{E}g(||X||)=a, \mathbb{E}g(||Y||)=b\},$$

where $a>0$, $b>0$, $h\in M$, $g\in M$. In particular, for all $p\geq 0$, $q\geq 0$ the values

$$(2.23)\quad I(p,q;a,b) := I(h,g;a,b)(h(t)=t^p,\ g(t)=t^q),$$

$$(2.4)\quad S(p,q;a,b) := S(h,g;a,b)(h(t)=t^p,\ g(t)=t^q)$$

are calculated. Note that here and in the sequel $\mathbb{E}||X-Y||^0$ means $\Pr(X\neq Y)$.

The scheme of the proofs of all statements here is as follows: first we prove the necessary inequalities that give us the required bounds and then we construct pairs of random variables which achieve the bounds or approximate them with arbitrary precision.

Let $f, f_1, f_2 \in M$ and consider the following conditions (here and in the sequel f^{-1} is the inverse function of $f\in M$):

$A(f_1,f_2)$: $f_1\circ f_2^{-1}(t)$ $(t\geq 0)$ is convex;

$B(f)$: $f^{-1}(\mathbb{E}f(||X+Y||)) \leq f^{-1}(\mathbb{E}f(||X||)) + f^{-1}(\mathbb{E}f(||Y||))$ for any $X,Y\in \mathfrak{X}$;

$C(f)$: $\mathbb{E}f(||X+Y||) \leq \mathbb{E}f(||X||) + \mathbb{E}f(||Y||)$ for any $X,Y\in\mathfrak{X}$;

$D(f_1,f_2)$: $\lim_{t\to\infty} f_1(t)/f_2(t) = 0$;

$E(f_1,f_2): f_1 \circ f_2(t)$ $(t \geq 0)$ is concave;

$F(f_1,f_2): f_1$ is concave and f_2 is convex;

$G(f_1,f_2): \lim_{t\to\infty} f_1(t)/f_2(t) = \infty$.

Obviously, if $h(t) = t^p$, $g(t) = t^q$ $(p > 0, q > 0)$, then $A(h,g) \Leftrightarrow p \geq q$, $B(g) \Leftrightarrow q \geq 1$, $C(g) \Leftrightarrow q \leq 1$, $D(h,g) \Leftrightarrow q > p$, $E(h,g) \Leftrightarrow q \geq p$, $F(h,g) \Leftrightarrow p \leq 1 \leq q$, $G(h,g) \Leftrightarrow p > q$ and hence, the conditions A-G cover all possible values of the pairs (p,q).

In the next theorem we establish explicit solutions of the moment problem (1.17) under the conditions (1.18) and combinations of requirements A-G on h and g.

Theorem 1. *For any* $a \geq 0$ *and* $b \geq 0$, $a + b > 0$ *the following equalities hold:*

i)

$$(2.5)\quad I(h,g;a,b) = \begin{cases} h(|g^{-1}(a) - g^{-1}(b)|) & \text{if } A(h,g) \text{ and } B(g) \text{ hold,} \\ h\circ g^{-1}(|\alpha - \beta|) & \text{if } A(h,g) \text{ and } C(g) \text{ hold,} \\ 0 & \text{if } D(h,g) \text{ holds.} \end{cases}$$

ii) *For any* $u \in U$, $h \in M$ *and* $g \in M$

$$(2.6)\quad \inf\{\Pr\{X \neq Y\}: \mathbb{E}g(\|X\|) = a, \mathbb{E}g(\|Y\|) = b\} = 0,$$

$$(2.7)\quad \inf\{\mathbb{E}h(\|X - Y\|): \Pr\{X \neq u\} = a, \Pr\{Y \neq u\} = b\} = 0 \quad (a,b \in [0,1]).$$

iii)

$$(2.8)\quad S(h,g;a,b) = \begin{cases} h(g^{-1}(a) + g^{-1}(b)) & \text{if } F(h,g) \text{ holds or if } B(g) \text{ and } E(h,g) \text{ hold,} \\ h\circ g^{-1}(\alpha + \beta) & \text{if } C(g) \text{ and } E(h,g) \text{ hold,} \\ \infty & \text{if } G(h,g) \text{ holds.} \end{cases}$$

iv) *For any* $u \in U$, $h \in M$, $g \in M$,

(2.9) $\sup\{\Pr\{X \neq Y\} : \mathbb{E}g(\|X\|) = a,\ \mathbb{E}g(\|Y\|) = b\} = 1,$

(2.10) $\sup\{\Pr\{X \neq Y\} : \Pr\{X \neq u\} = a,\ \Pr\{Y \neq u\} = b\} = \min(a+b, 1) \quad (a, b \in [0,1]),$

(2.11) $\sup\{\mathbb{E}h(\|X-Y\|) : \Pr(X \neq u) = a,\ \Pr\{X \neq u\} = b\} = \infty.$

<u>Proof</u>: i) *Case 1.* Let $A(h,g)$ and $B(g)$ be fulfilled. Denote $\phi(a,b) := h(|g^{-1}(a) - g^{-1}(b)|)$, $a \geq 0$, $b \geq 0$.

<u>Claim 1</u>. $I(h,g,a,b) \geq \phi(a,b)$. By the Jensen's inequality and $A(h,g)$

(2.12) $h \circ g^{-1}(\mathbb{E}Z) \leq \mathbb{E}h \circ g^{-1}(Z).$

Taking $Z = g(\|X - Y\|)$ and using $B(g)$ we obtain

$$h^{-1}(\mathbb{E}h(\|X-Y\|)) \geq g^{-1}(\mathbb{E}g(\|X-Y\|)) \geq |g^{-1}(\mathbb{E}g(\|X\|)) - g^{-1}(\mathbb{E}g(\|Y\|))|$$

for any $X, Y \in X$ which proves the claim.

<u>Claim 2</u>. There exists an "optimal" pair (X^*, Y^*) of rv's such that $\mathbb{E}g(\|X^*\|) = a$, $\mathbb{E}g(\|Y^*\|) = b$, $\mathbb{E}(\|X^* - Y^*\|) = \phi(a,b)$. Let $\bar{e}$ here and in the sequel be a fixed point of U with $\|\bar{e}\| = 1$. Then the required pair (X^*, Y^*) is given by

(2.13) $X^* = g^{-1}(a)\bar{e},\ Y^* = g^{-1}(b)\bar{e}$

which proves the claim.

Case 2. Let $A(h,g)$ and $C(g)$ be fulfilled. Denote $\phi_1(t) := h \circ g^{-1}(t)$, $t \geq 0$. As in Claim 1 we get $I(h,g;a,b) \geq \phi_1(|a-b|)$. Suppose that $a > b$ and for each $\varepsilon > 0$ define a pair $(X_\varepsilon, Y_\varepsilon)$ of rv's as follows:

$$\Pr\{X_\varepsilon = c_\varepsilon \bar{e},\ Y_\varepsilon = \bar{0}\} = p_\varepsilon,\ \Pr\{X_\varepsilon = d_\varepsilon \bar{e},\ Y_\varepsilon = d_\varepsilon \bar{e}\} = 1 - p_\varepsilon,$$

where

(2.14) $\bar{0} := 0\bar{e}$, $p_\varepsilon := \frac{a-b}{a-b+\varepsilon}$, $c_\varepsilon := g^{-1}(a-b+\varepsilon)$, $d_\varepsilon := g^{-1}(\frac{b}{1-p_\varepsilon})$.

Then $(X_\varepsilon, Y_\varepsilon)$ enjoys the side conditions in (2.1) and

$$\mathbb{E}h(\|X_\varepsilon - Y_\varepsilon\|) = \phi_1(a-b+\varepsilon)\frac{a-b}{a-b+\varepsilon}.$$

Letting $\varepsilon \to 0$ we claim (2.5).

Case 3. Let $D(h,g)$ be fulfilled. In order to obtain (2.5) it is sufficient to define a sequence (X_n, Y_n) $(n \geq N)$ such that $\lim_{n\to\infty} \mathbb{E}\, h(\|X_n - Y_n\|) = 0$, $\mathbb{E}g(\|X_n\|) = a$, $\mathbb{E}g(\|Y_n\|) = b$. An example of such sequence is the following one:

$$\Pr\{X_n = \bar{0},\ Y_n = \bar{0}\} = 1 - c_n - d_n,$$

$$\Pr\{X_n = na\bar{e},\ Y_n = \bar{0}\} = c_n,$$

$$\Pr\{X_n = \bar{0},\ Y_n = nb\bar{e}\} = d_n,$$

where $c_n = a/g(na)$, $d_n = b/g(nb)$ and N satisfies $c_N + d_N < 1$.

ii) Define the sequence (X_n, Y_n) $(n = 2,3,\ldots)$ such that

$$\Pr\{X_n = g^{-1}(na)\bar{e},\ Y_n = g^{-1}(nb)\} = \frac{1}{n},$$

$$\Pr\{X_n = \bar{0},\ Y_n = \bar{0}\} = \frac{n-1}{n}.$$

Hence, $\mathbb{E}g(\|X_n\|) = a$, $\mathbb{E}g(\|Y_n\|) = b$ and $\Pr(X_n \neq Y_n) = \frac{1}{n}$ which shows (2.6).

Further suppose $a \geq b$. Without loss of generality we may assume that $u = \bar{0}$. Then consider the random pair $(\tilde{X}_n, \tilde{Y}_n)$ with the following joint distribution:

$\Pr\{\tilde{X}_n = \bar{0},\ \tilde{Y}_n = \bar{0}\} = 1-a$, $\Pr\{\tilde{X}_n = \frac{1}{n}\bar{e},\ \tilde{Y}_n = \bar{0}\} = a-b$, $\Pr\{\tilde{X}_n = \frac{1}{n}\bar{e},\ \tilde{Y}_n = \frac{1}{n}\bar{e}\} = b$.

Obviously, $(\tilde{X}_n, \tilde{Y}_n)$ satisfies the side conditions

$$\Pr(\tilde{X}_n \neq \bar{0}) = a,\ \Pr(\tilde{Y}_n \neq \bar{0}) = b \quad \text{and} \quad \lim_{n\to\infty} \mathbb{E}\, h(\|X_n - Y_n\|) = 0,$$

which proves (2.7).

The proofs of iii) and iv) are quite analogous to those of i) and ii) respectively. QED

Note that if $A(h,g)$ and $B(g)$ hold we have constructed an *optimal* pair (X^*,Y^*) (see (2.13)), i.e. (X^*,Y^*) realizes the infimum in (2.1). However, if $D(h,g)$ holds and $a \neq b$ then optimal pairs do not exist, because $\mathbb{E}h(\|X-Y\|) = 0$ implies $a = b$.

Corollary 1. *For any* $a \geq 0$, $b \geq 0$, $a+b \geq 0$, $p \geq 0$, $q \geq 0$,

$$(2.15) \quad I(p,q;a,b) = \begin{cases} |a^{1/q} - b^{1/q}|^p & \textit{if } p \geq q \geq 1, \\ |a-b|^{p/q} & \textit{if } p \geq q,\ 0 < q < 1, \\ 0 & \textit{if } 0 \leq p < q \textit{ or } q = 0,\ p > 0, \\ |a-b| & \textit{if } p = q = 0. \end{cases}$$

$$(2.16) \quad S(p,q;a,b) = \begin{cases} (a^{1/q} + b^{1/q})^p & \textit{if } 0 \leq p \leq q,\ q \geq 1, \\ (a+b)^{p/q} & \textit{if } 0 \leq p \leq q < 1,\ q \neq 0, \\ \infty & \textit{if } p > q \geq 0, \\ \min(a+b,1) & \textit{if } p = q = 0. \end{cases}$$

Remark. Obviously, if $q = 0$ in (2.15) or (2.16) the values of I and S make sense for $a,b \in [0,1]$.

3. Moment problems with two fixed pairs of marginal moments

The main part of this section is devoted to the explicit description of the following bounds:

$$(3.1)\quad I(h,g_1,g_2;a_1,b_1,a_2,b_2) := \inf \mathbb{E}\, h(\|X-Y\|),$$

$$(3.2)\quad S(h,g_1,g_2;a_1,b_1,a_2,b_2) := \sup \mathbb{E}\, h(\|X-Y\|),$$

where $h,g_1,g_2 \in M$, and the infimum in (3.1) and the supremum in (3.2) are taken over the set of all pairs of rv's $X,Y \in X(U)$ satisfying the moment conditions

$$(3.3)\quad \mathbb{E}\, g_i(\|X\|) = a_i,\quad \mathbb{E}\, g_i(\|Y\|) = g_i,\ i=1,2,.$$

In particular, if $h(t)=t^p$, $g_i(t)=t^{q_i}$, $i=1,2$ $(p \geq 0,\ q_2 > q_1 \geq 0)$, we write

$$(3.4)\quad I(p,q_1,q_2;a_1,b_1,a_2,b_2) := I(h,g_1,g_2;a_1,b_1,a_2,b_2),$$

$$(3.5)\quad S(p,q_1,q_2;a_1,b_1,a_2,b_2) := S(h,g_1,g_2;a_1,b_1,a_2,b_2).$$

The moment problem with two pairs of marginal conditions is considerably more complicated and in the present section, our results are not as complete as in the previous one.

Theorem 2. *Let the conditions* $A(g_2,g_1)$ *and* $G(g_2,g_1)$ *hold. Let* $a_i \geq 0$, $b_i \geq 0$, $i=1,2$, $a_1+a_2 > 0$, $b_1+b_2 > 0$ *and*

$$(3.6)\quad g_1^{-1}(a_1) \leq g_2^{-1}(a_2),\quad g_1^{-1}(b_1) \leq g_2^{-1}(b_2).$$

i) *If* $A(h,g_1)$, $B(g_1)$ *and* $D(h,g_2)$ *are fulfilled, then*

$$(3.7)\quad I(h,g_1,g_2;a_1,b_1,a_2,b_2) = I(h,g_1;a_1,b_1) = h(|g_1^{-1}(a_1) - g_1^{-1}(b_1)|).$$

ii) *Let* $D(h,g_2)$ *be fulfilled. If* $F(h,g)$ *holds or if* $B(g)$ *and* $E(h,g)$ *hold then*

$$(3.8)\quad S(h,g_1,g_2;a_1,b_1,a_2,b_2) = S(h,g_1;a_1,b_1) = h(g_1^{-1}(a_1)+g_1^{-1}(b_1)).$$

iii) *If* $G(h,g_2)$ *is fulfilled and* $g_1^{-1}(a_1) \neq g_2^{-1}(a_2)$ *or* $g_1^{-1}(b_1) \neq g_2^{-1}(b_2)$, *then*

$$(3.9)\quad S(h,g_1,g_2;a_1,b_1,a_2,b_2) = S(h,g_1;a_1,b_1) = \infty.$$

Proof: By Theorem 1 i) we have

$$(3.10)\quad I(h,g_1,g_2;a_1,b_1,a_2,b_2) \geq I(h,g_1,a_1,b_1) = \phi(a_1,b_1).$$

Further we shall define an appropriate sequence of rv's (X_t,Y_t) that satisfy the side conditions (3.3) and $\lim_{t\to\infty} \mathbb{E}\, h(\|X_t - Y_t\|) = \phi(a_1,b_1)$. Let $f(x) = g_2 \circ g_1^{-1}(x)$. Then, by the Jensen's inequality and $A(g_2,g_1)$,

$$(3.11)\quad f(a_1) = f(\mathbb{E} g_1(\|X\|)) \leq \mathbb{E} f\circ g_1(\|X\|) = a_2$$

as well as $f(b_1) \leq b_2$. Moreover $\lim_{t\to\infty} f(t)/t = \infty$ by $G(g_1,g_2)$.

Case 1. Suppose that $f(a_1) < a_2$, $f(b_1) < b_2$.

Claim. If the functions $f \in M$ and the reals c_1, c_2 possess the properties

$$(3.12)\quad f(c_1) < c_2,\ \lim_{t\to\infty} f(t)/t = \infty$$

then there exist a positive t_0 and a function $k(t)$ $(t \geq t_0)$ such that the following relations hold for any $t \geq t_0$:

$$(3.13)\quad 0 < k(t) < c_1,$$

$$(3.14)\quad tf(c_1 - k(t)) + k(t)f(c_1+t) = c_2(k(t)+t),$$

$$(3.15) \quad \frac{k(t)}{k(t)+t} \le \frac{c_2}{f(c_1+t)} ,$$

and

$$(3.16) \quad \lim_{t\to\infty} k(t) = 0.$$

Proof of the claim: Let us take such t_0 that $f(c_1+t)/(c_1+t) > c_2/c_1$, $t \ge t_0$ and consider the equation

$$F(t,x) = c_2,$$

where $F(t,x) := f(c_1 - x)t/(x+t) + f(c_1+t)x/(x+t)$. For each $t \ge t_0$ we have $F(t,c_1) > c_2$, $F(t,0) = f(c_1) < c_2$. Hence, for each $t \ge t_0$ there exists such $x = k(t)$, that $k(t) \in (0,c_1)$ and $F(t,k(t)) = c_2$, which provides (3.13) and (3.14). Further (3.14) implies (3.15) as well as (3.13) and (3.15) imply (3.16). The claim is established.

From the claim, we see that there exist $t_0 > 0$ and functions $\ell(t)$ and $m(t)$ $(t \ge t_0)$ such that for all $t > t_0$ we have

$$(3.17) \quad 0 < \ell(t) < a_1, \; 0 < m(t) < b_1,$$

$$(3.18) \quad tf(a_1 - \ell(t)) + \ell(t)f(a_1+t) = a_2(\ell(t)+t),$$

$$(3.19) \quad tf(b_1 - m(t)) + m(t)f(b_1+t) = b_2(m(t)+t),$$

$$(3.20) \quad \lim_{t\to\infty} \ell(t) = 0, \; \lim_{t\to\infty} m(t) = 0.$$

Using (3.17)-(3.20) and the conditions $A(h,g_1)$, $D(h,g_2)$ and $G(g_2,g_1)$, one can easily check that the rv's (X_t,Y_t) $(t > t_0)$ determined by the equalities

$$\Pr\{X_t = x_i(t),\; Y_t = y_j(t)\} = p_{ij}(t),\; i,j = 1,2,$$

where

$$x_1(t) := g_1^{-1}(a_1 - \ell(t))\bar{e},\ x_2(t) := g_1^{-1}(a_1 + t)\bar{e},$$

$$y_1(t) := g_1^{-1}(b_1 - m(t))\bar{e},\ y_2(t) := g_1^{-1}(b_1 + t)\bar{e},$$

$$p_{11}(t) := \min\{t/(\ell(t) + t),\ t/(m(t) + t)\},$$

$$p_{12}(t) := t/(\ell(t) + t) - p_{11}(t),$$

$$p_{21}(t) := t/(m(t) + t) - p_{11}(t),$$

$$p_{22}(t) := \min\{\ell(t)/(\ell(t) + t),\ m(t)/(m(t) + t)\}$$

possess all the desired properties.

Case 2. Suppose $f(a_1) = a_2$ (i.e. $g_1^{-1}(a_1) = g_2^{-1}(a_2)$), $f(b_1) < b_2$. Then we determine (X_t, Y_t) by the equalities

$$\Pr\{X_t = g_1^{-1}(a_1),\ Y_t = y_1(t)\} = t/(m(t) + t),$$

$$\Pr\{X_t = g_1^{-1}(a_1),\ Y_t = y_2(t)\} = m(t)/(m(t) + t).$$

Case 3. The cases $(f(a_1) < a_2,\ f(b_1) = b_2)$, $(f(a_1) = a_2,\ f(b_1) = b_2)$ are considered in the same way as in Case 2.

ii) and iii) are proved by the analogous arguments. QED

Corollary 2. *Let* $a_i \ge 0$, $b_i \ge 0$, $a_1 + a_2 > 0$, $b_1 + b_2 > 0$ $a_1^{1/q_1} \le a_2^{1/q_2}$, $b_1^{1/q_1} \le b_2^{1/q_2}$.

i) *If* $1 \le q_1 \le p < q_2$, *then*

$$I(p, q_1, q_2; a_1, b_1, a_2, b_2) = I(p, q_1; a_1, b_1) = (a_1^{1/q_1} - b_1^{1/q_1})^p. \tag{3.21}$$

ii) *If* $0 < p \le q_1$, $1 \le q_1 < q_2$ *then*

(3.22) $S(p,q_1,q_2;a_1,b_1,a_2,b_2) = S(p,q_1;a_1,b_1) = (a_1^{1/q_1} + b_1^{1/q_1})^p$.

iii) *If* $0 < q_1 < q_2 < p$ *and* $a_1^{1/q_1} = a_2^{1/q_2}$ *or* $b_1^{1/q_1} = b_2^{1/q_2}$ *then*

$$S(p,q_1,q_2;a_1,b_1,a_2,b_2) = S(p,q_1;a_1,b_1) = \infty.$$

Corollary 2 describes situations in which the "additional moment information" $a_2 = \mathbb{E}||X||^{q_2}$, $b_2 = \mathbb{E}||Y||^{q_2}$ does not affect the bounds

$I(p,q_1,q_2;a_1,b_1,a_2,b_2) = I(p,q_1,a_1,a_2)$, $S(p,q_1,q_2;a_1,b_1,a_2,b_2) = S(p,q_1,a_1,a_2)$, (and likewise Theorem 2).

We conclude this section by giving the explicit solution of the following moment problem: *determine the extremal values*

(3.23) $\tilde{I} := \tilde{I}(p,q_1,q_2,a_1,a_2) := \inf\{\mathbb{E}||X||^p : \mathbb{E}||X||^{q_1} = a_1,\ \mathbb{E}||X||^{q_2} = a_2\}$,

(3.24) $\tilde{S} := \tilde{S}(p,q_1,q_2,a_1,a_2) := \sup\{\mathbb{E}||X||^p : \mathbb{E}||X||^{q_1} = a_1,\ \mathbb{E}||X||^{q_2} = a_2\}$

for all $p \ge 0$, $0 \le q_1 \le q_2$.

<u>Theorem</u> 3. *Let* $p \ge 0$, $0 \le q_1 < q_2$. *Then*

$$\tilde{I} = \begin{cases} \dfrac{a_1^{(q_2-p)/(q_2-q_1)}}{a_2^{(q_1-p)/(q_2-q_1)}} & \text{if } 0 \le p \le q_1 < q_2, \\[2ex] \dfrac{a_2^{(p-q_1)/(q_2-q_1)}}{a_1^{(p-q_2)/(q_2-q_1)}} & \text{if } 0 \le q_1 < q_2 \le p, \\[2ex] a_1^{p/q} & \text{if } 0 < q_1 \le p < q_2 \text{ or } a_1^{1/q_1} = a_2^{1/q_2},\ 0 < q_1 < q_2, \\ a_i & \text{if } p = q_i \quad i = 1 \text{ or } 2 \\ 0 & \text{if } 0 = q_1 < p < q_2 \end{cases}$$

and

$$\tilde{S} = \begin{cases} \dfrac{a_1^{(q_2-p)/(q_2-q_1)}}{a_2^{(q_1-p)/(q_2-q_1)}} & \text{if } 0 \le q_1 \le p \le q_2,\ q_1 < q_2 \\ a_1^{p/q_1} & \text{if } 0 < p \le q_1 < q_2 \text{ or } 0 = p < q_1 < q_2 \\ & \text{or } a_1^{1/q_1} = a_2^{1/q_2},\ 0 < q_1 < q_2, \\ a_i & \text{if } p = q_i,\ i = 1 \text{ or } 2, \\ \infty & \text{if } 0 \le q_1 < q_2 < p,\ a_1^{1/q_1} \ne a_2^{1/q_2} \\ 1 & \text{if } p = 0,\ 0 < q_1 < q_2 \text{ or } a_1^{1/q_1} = a_2^{1/q_2}. \end{cases}$$

Proof: If $a_1^{1/q_1} = a_2^{1/q_2}$ then it is easy to see that $||X|| = a_1^{1/q_1}$ with probability 1. Hence,

$$\tilde{I} = \tilde{S} = a_1^{p/q_1} \text{ if } p > 0,\ 0 < q_1 < q_2,\ a_1^{1/q_1} = a_2^{1/q_2}. \tag{3.25}$$

Further, suppose that $a_1^{1/q_1} \ne a_2^{1/q_2}$. For any $r > 1$ and any real-valued ξ and η,

$$\mathbb{E}|\xi\eta| \le (\mathbb{E}|\xi|^r)^{1/r}(\mathbb{E}|\eta|^{\frac{r}{r-1}})^{\frac{r-1}{r}} \tag{3.26}$$

by the Hölder's inequality. Let $0 < r_1 < r_2 < r_3$, $r = (r_3 - r_1)/(r_3 - r_2)$, $\xi = ||X||^{r_1(r_3-r_2)/(r_3-r_1)}$, $\eta = ||X||^{r_3(r_2-r_1)/(r_3-r_2)}$. Then by (3.26),

$$\mathbb{E}||X||^{r_2} \le (\mathbb{E}||X||^{r_1})^{(r_3-r_2)/(r_3-r_1)}(\mathbb{E}||X||^{r_3})^{(r_2-r_1)/(r_3-r_2)}. \tag{3.27}$$

Taking $r_1 = p$, $r_2 = q_1$, $r_3 = q_2$ in (3.37) we obtain

$$(3.28)\qquad \mathbb{E}\|X\|^{q_1} \le (\mathbb{E}\|X\|^{p})^{(q_2 - q_1)/(q_2 - p)} (\mathbb{E}\|X\|^{q_2})^{(q_1 - p)/(q_2 - p)}$$

which implies for $0 < q_1 < q_2$,

$$(3.29)\qquad \tilde{I} \ge a_1^{(q_2 - p)/(q_2 - q_1)} a_2^{(p - q_1)/(q_2 - q_1)}.$$

Taking $r_1 = q_1$, $r_2 = q_2$, $r_3 = p$ in (3. 7) we obtain

$$(3.30)\qquad \tilde{I} \ge a_2^{(p - q_1)/(q_2 - q_1)} a_1^{(p - q_2)/(q_2 - q_1)} \quad \text{for } q_1 < q_2 < p.$$

Finally, putting $r_1 = q_1$, $r_2 = p$, $r_3 = q_2$ in (3.27) we have

$$(3.31)\qquad \tilde{S} \le a_1^{(q_2 - p)/(q_2 - q_1)} a_2^{(p - q_1)/(q_2 - q_1)} \quad \text{for } q_1 < p < q_2.$$

The "optimal" rv X^* (for all inequalities (3.29)-(3.31) and $p \neq 0$) is given by

$$\Pr\{X^* = a_2^{1/(q_2 - q_1)} a_1^{-1/(q_2 - q_1)} \bar{e}\} = a_1^{q_2/(q_2 - q_1)} a_2^{-q_1/(q_2 - q_1)} := b,$$

$$\Pr\{X^* = 0\} = 1 - b.$$

Thus the equalities

$$(3.32)\qquad \tilde{I} = \begin{cases} a_1^{(q_2 - p)/(q_2 - q_1)} a_2^{-(q_1 - p)/(q_2 - q_1)} & \text{if } 0 < p < q_1 < q_2 \\ a_2^{(p - q_1)/(q_2 - q_1)} a_1^{-(p - q_2)/(q_2 - q_1)} & \text{if } 0 < q_1 < q_2 < p, \end{cases}$$

$$(3.33)\qquad \tilde{S} = a_1^{(q_2 - p)/(q_2 - q_1)} a_2^{(p - q_1)/(q_2 - q_1)} \quad \text{if } 0 < q_1 < p < q_2$$

and the inequality

$$(3.34)\quad \tilde{I} \le a_1^{q_2/(q_2-q_1)} a_2^{-q_1/(q_2-q_1)} \qquad \text{if } p=0,\ q_1<q_2$$

are claimed.

Further, Ljapunov's inequality implies

$$(3.35)\quad \tilde{I} \ge a_1^{p/q_1} \quad \text{for } q_1<p$$

and

$$(3.36)\quad \tilde{S} \le a_1^{p/q_1} \quad \text{for } p<q_1.$$

Thus, it is sufficient to determine a sequence of rv's $\{X_t\}_{t\ge t_0}$ such that

$$(3.37)\quad \mathbb{E}\|X_t\|^{q_1} = a_1,\ \mathbb{E}\|X_t\|^{q_2} = a_2,\ \lim_{t\to\infty}\mathbb{E}\|X_t\|^{p} = a_1^{p/q_1}.$$

Now we can use the claim in the proof of Theorem 2 with $f(t)=t^{q_2/q_1}$, $c_i=a_i$ $(i=1,2)$ and define the sequence $\{X_t\}_{t\ge t_0}$ by

$$(3.38)\quad \Pr\{X_t = (a_1-k(t))^{1/q_1}\bar{e}\} = \frac{t}{k(t)+t}$$

$$(3.39)\quad \Pr\{X_t = (a_1+t)^{1/q_1}\bar{e}\} = \frac{k(t)}{k(t)+t},$$

where $k(t)$ satisfies the relationships (3.13)-(3.16). By (3.35)-(3.37), it follows that

$$(3.40)\quad \tilde{I} = a_1^{p/q_1} \quad \text{if } 0<q_1<p<q_2,$$

$$(3.41)\quad \tilde{S} = a_1^{p/q_1} \quad \text{if } 0<p<q_1<q_2.$$

If $p > q_2$ then the sequence $\{X_t\}$, $t \geq 0$, possesses the property $\lim_{t\to\infty} \mathbb{E}||X_t||^p = \infty$. So,

(3.42) $\tilde{S} = \infty$ if $0 < q_1 < q_2 < p$.

Putting $r = r_2/r_1$, $\xi = ||X||^{r_1}$, $\eta = I\{X \neq \bar{0}\}$ in (3. 6) we obtain

(3.43) $\mathbb{E}||X||^{r_1} \leq (\mathbb{E}||X||^{r_2})^{r_1/r_2}(\Pr\{X \neq \bar{0}\})^{(r_2 - r_1)/r_2}$

Taking $r_1 = q_1$, $r_2 = q_2$ we have

(3.44) $\tilde{I} \geq a_1^{q_2/(q_2 - q_1)} a_2^{-q_1/(q_2 - q_1)}$ for $0 = p < q_1 < q_2$.

By (3.34) and (3.43),

(3.45) $\tilde{I} = a_1^{q_2/(q_2 - q_1)} a_2^{-q_1/(q_2 - q_1)}$ for $0 = p < q_1 < q_2$.

Analogously, letting $r_1 = p$, $r_2 = q_2$ in (3.43) we have

(3.46) $\tilde{S} \leq a_1^{(q_2 - p)/q_2} a_2^{p/q_2}$ if $0 = q_1 < q_2 < p$.

Also (3. 3) with $r_1 = q_2$, $r_2 = p$ implies

(3.47) $\tilde{I} \geq a_2^{p/q_2} a_1^{(p - q_2)/q_2}$ if $0 = q_1 < q_2 < p$.

The "optimal" rv X* for the last two cases is given by

$$\Pr\{X^* = (\frac{a_2}{a_1})^{1/q_2}\} = a_1, \ \Pr\{X^* = 0\} = 1 - a.$$

Thus,

(3.48) $\tilde{S} = a_1^{(q_2-p)/q_2} a_2^{p/q_2}$, $\tilde{I} = a_2^{p/q_2} a_1^{(p-q_2)/q_2}$ if $0 = q_1 < q_2 < p$.

Further, the sequence $\{X_n\}_{n>1/a_1}$ defined by

$$\Pr\{X_n = (na_2 - 1)^{1/q_2}\bar{e}\} = 1/n,$$

$$\Pr\{X_n = (na_2 - 1)^{-1/q_2}\bar{e}\} = a_1 - 1/n,$$

$$\Pr\{X_n = \bar{0}\} = 1 - a_1$$

possesses the properties:

$$\Pr\{X_n \neq \bar{0}\} = a_1, \quad \mathbb{E}\|X_n\|^{q_2} = a_2,$$

$$\lim_{n\to\infty} \mathbb{E}\|X_n\|^p = \infty \quad \text{if } q_2 < p,$$

$$\lim_{n\to\infty} \mathbb{E}\|X_n\|^p = 0 \quad \text{if } p < q_2.$$

Hence,

(3.49) $\tilde{S} = \infty$ if $0 = q_1 < q_2 < p$,

(3.50) $\tilde{I} = 0$ if $0 = q_1 < p < q_2$.

Finally, if $p = q_i$, then

(3.51) $\tilde{I} = \tilde{S} = a_i$, $i = 1, 2$.

Summarizing (3.32), (3.32), (3.40), (3.41), (3.42), (3.45), (3.48)-(3.51) we obtain the desired explicit representations for $\tilde{I}$ and $\tilde{S}$.

4. General remarks

Combining Theorems A and 1 we obtain the following precise estimates of the extremal functionals $\hat{\ell}_H(P,Q)$ $(P,Q \in \mathcal{P}(U))$ and $\check{\ell}_H(P,Q)$ (see Theorem A) in terms of the moments

$$(4.1) \quad a = \int_U g(x)P(dx), \ b = \int_U g(x)Q(dx).$$

Theorem 4. *Let* $(U, \|\cdot\|)$ *be a separable normed space and* $H \in \mathcal{H}$ (see (1.7)).

i) *If* $A(H,g)$ *and* $B(g)$ *hold, then*

$$(4.2) \quad \hat{\ell}_H(P,Q) \geq H(|g^{-1}(a) - g^{-1}(b)|).$$

ii) *If* $B(g)$ *and* $E(H,g)$ *hold, then*

$$(4.3) \quad \check{\ell}_H(P,Q) \leq H(g^{-1}(a) + g^{-1}(b)).$$

Moreover there exist P_i, $Q_i \in \mathcal{P}(U)$, $i = 1,2$ with

$$a = \int_U g(x)P_i(dx), \ b = \int_U g(x)Q_i(dx)$$

such that

$$\hat{\ell}_H(P_1,Q_1) = H(|g^{-1}(a) - g^{-1}(b)|)$$

and

$$\check{\ell}_H(P_2,Q_2) = H(g^{-1}(a) + g^{-1}(b)).$$

The following theorem is an extension for $p = q = 1$ of Theorem 1 to a non-normed space U such as the Skorokhod space D[0,1], which is of special interest in probability and statistics (see Billingsley (1968)).

Theorem 5. *Let* (U,d) *be a separable metric space,* $\mathcal{X} = \mathcal{X}(U)$ *the space of all* U*-valued rv's,* $u \in U$, $a \geq 0$, $b \geq 0$. *Assume that there exists* $z \in U$ *such that* $d(z,u) \geq \max(a,b)$. *Then*

$$(4.4) \quad \min\{\mathbb{E}\, d(X,Y) : X,Y \in \mathcal{X},\ \mathbb{E} d(X,u) = a,\ \mathbb{E}\, d(Y,u) = b\} = |a-b|$$

and

$$(4.5) \quad \max\{\mathbb{E}\, d(X,Y) : X,Y \in \mathcal{X},\ \mathbb{E} d(X,u) = a,\ \mathbb{E}\, d(Y,u) = b\} = a+b.$$

Proof: Let $a \leq b$, $\gamma = d(z,u)$. By the triangle inequality the minimum in (4.4) is greater than $b-a$. On the other hand if $\Pr(X=u,\ Y=u) = 1-b/\gamma$, $\Pr(X=u,\ Y=z) = (b-a)/\gamma$, $\Pr(Z=z,\ Y=u) = 0$, $\Pr(X=z,\ Y=z) = a/\gamma$ then $\mathbb{E} d(X,u) = a$, $\mathbb{E} d(Y,u) = b$, $\mathbb{E}\, d(X,Y) = b-a$, which proves (4.4). Analogously, one shows (4.5). QED

We conclude by stating explicitly the following open problems.

(i) *Find the explicit expression of* $I(a,q_1,q_2;a_1,b_1,a_2,b_2)$ *and* $S(p,q_1,q_2;a_1,b_1,a_2,b_2)$ *for all* $p \geq 0$, $q_2 > 0$, $q_1 \geq 0$ (see (3.4), Corollary 2 and Theorem 3).

One could start with the following one-dimensional version of i). Let $g_i : [0,\infty) \to \mathbb{R}$ $(i=1,2)$ and $h : \mathbb{R} \to \mathbb{R}$ be given continuous functions with h symmetric and strictly increasing on $[0,\infty)$. Let further X and Y be nonnegative random variables having given moments

$$a_i = \mathbb{E}\, g_i(x),\ b_i = \mathbb{E}\, g_i(Y),\ i=1,2.$$

The problem is to evaluate

$$(4.5) \quad I = \inf \mathbb{E} h(X-Y),\ S = \sup \mathbb{E}\, h(X+Y).$$

If desired one could think of $X = X(t)$ and $Y = Y(t)$ as functions on the unit interval (with Lebesgue measure), see Karlin and Studden (1966), Chapter 3 and Rogosinski (1958).

The 5 moments a_1, a_2, b_1, b_2 and $\mathbb{E}h(X \pm Y)$ depend only on the joint distribution of the pair (X,Y) and the extremal values in (4.5) are realized by a probability measure supported by 6 points. (See Rogosinski (1958), Theorem 1; Karlin and Studden (1966), Chapter 3; Kemperman (1985)). Thus the problem can also be formulated as the nonlinear programming problem to find

$$I = \inf \sum_{j=1}^{6} p_j h(u_j - v_j), \quad S = \sup \sum_{j=1}^{6} p_j h(u_j + v_j),$$

subject to

$$p_j \geq 0, \quad \sum_{j=1}^{6} p_j = 1, \quad u_j \geq 0, \quad v_j \geq 0, \quad j = 1, \ldots, 6,$$

$$\sum_{j=1}^{6} p_j g_i(u_j) = a_i, \quad \sum_{j=1}^{6} p_j g_i(v_j) = b_i, \quad i = 1, 2, .$$

Such a problem becomes simpler when the functions g_i and h on R_+ are convex (see, for example, Karlin and Studden (1966), Chapter XIV).

(ii) *Find the explicit solutions of moment problems with one fixed pair of marginal moments for rv's with values in a separable metric space* U. (See Theorem 5).

Note that in the case when U is a normed space, the moment problem was easily reduced to the one-dimensional moment problem ($U = \mathbb{R}$). This is no longer possible for general (non-normed) spaces U rendering the moment problem (ii) quite different for that considered in Sections 2 and 3.

Acknowledgements. We wish to express our gratitude to S. Cambanis, J.H.B. Kemperman, G. Simons and J.C. Smit for their helpful comments and suggestions.

References

Ahiezer, N.I. and Krein, M. (1962). Some Questions in the Theory of Moments. American Mathematical Society, Providence.

Basu, S.K. and Simons, G. (1983). Moment spaces for IFR distributions, applications, and related material. Contributions to Statistics: Essays in Honour of Norman L. Johnson, P.K. Sen (Ed.), North-Holland Publishing Company, 27-46.

Billingsley, P. (1968). Convergence of Probability Measures. John Wiley and Sons, New York.

Cambanis, S. and Simons, G. (1982). Probability and expectation inequalities. Z. Wahrsch. verw. Geb. 59, 1-25.

Cambanis, S., Simons, G., and Stout, W. (1976). Inequalities for $\mathbb{E}k(X,Y)$ when the marginals are fixed. Z. Wahrsch. verw. Geb. 36, 285-294.

Haneveld, W.K.K. (1985). Duality in Stochastic Linear and Dynamic Programming. Centrum voor Wiskunde en Informatica, Amsterdam.

Hoeffding, W. (1955). The extrema of the expected value of a function of independent random variables. Ann. Math. Stat. 26, 268-275.

Hoeffding, W. and Shrikhande,S.S. (1956). Bounds for the distribution function of a sum of independent, identically distributed random variables. Ann. Math. Stat. 27, 439-449.

Karlin, S. and Studden, W.J. (1966). Tchebycheff Systems with Applications in Analysis and Statistics. Interscience Publishers, Wiley, New York.

Kellerer, H.G. (1984). Duality theorems for marginal problems. Z. Wahrsch. verw. Geb. 67, 399-432.

Kemperman, J.M.B. (1972). On a class of moment problems. Proceedings Sixth Berkeley Symposium on Mathematical Statistics and Probability 2, 101-126.

Kemperman, J.M.B. (1983). On the role of duality in the theory of moments. A.V. Fiacco, K.O. Kortanek (Eds.). Semi-Infinite Programming and Applications. Lecture Notes in Economics and Mathematical Systems 215, Springer, Berlin-New York, 63-92.

Levin, V.L. and Miljutin, A.A. (1978). The problem of mass transfer with a discontinuous cost function and a mass statement of the duality problem for convex extremal problems. Russian Math. Surveys 34/3, 1-78.

Rachev, S.T. (L984). The Monge-Kantorovich mass transference problem and its stochastic applications. Theory Probab. Appl. 29, 647-676.

Rachev, S.T. (1985). Extreme functionals in the space of probability measures. V.V. Kalashnikov, V.M. Zolotarev (Eds.). Stability Problems for Stochastic Models. Lecture Notes in Mathematics 1155, Springer, Berlin-New YOrk, 320-348.

Rachev, S.T. and Shortt, R.M. (1987). Duality theorems for Kantorovich-Rubinstein and Wasserstein functionals. Submitted to J. of Theoretical Probability.

Rogosinsky, W.W. (1958). Moments of non-negative mass. Proceedings Royal Society London, Ser. A. 245, 1-27.

Rüschendorf, L. (1985). The Wasserstein distance and approximation theorems. Z. Wahrsch. verw. Geb. 70, 117-129.

Shohat, J.A. and Tamarkin, J.D. (1943). The Problems of Moments. American Mathematical Society, Providence.

Sholpo, I.A. (L983). ε-minimal metrics. Theory Probab. Appl. 28, 854-855.

Zolotarev, V.M. (1976). Metric distances in spaces of random variables and their distributions. Math. USSR . 30/3, 373-401.

Zolotarev, V.M. (1983). Probability metrics. Theory Probab. Appl. 28, 278-302.

Department of Mathematics and Mechanics
Saratov State University
Astrakhanskaja str. 83
410601 Saratov, USSR

Institute of Mathematics
Bulgarian Academy of Sciences

and

Center for Stochastic Processes
Statistics Department
University of North Carolina
Chapel Hill, NC 27514
USA

END
DATE
FILMED
JAN
1988